



TITLE:

Singular solutions of q -difference-differential equations of Briot-Bouquet type
(New development of microlocal analysis
and singular perturbation theory)

AUTHOR(S):

Yamazawa, Hiroshi

CITATION:

Yamazawa, Hiroshi. Singular solutions of q -difference-differential equations of Briot-Bouquet type (New development of microlocal analysis and singular perturbation theory). 数理解析研究所講義録別冊 2019, B75: 73-87

ISSUE DATE:

2019-06

URL:

<http://hdl.handle.net/2433/244772>

RIGHT:

© 2019 by the Research Institute for Mathematical Sciences, Kyoto University. All rights reserved.

Singular solutions of q -difference-differential equations of Briot-Bouquet type

By

Hiroshi YAMAZAWA*

Abstract

In 1990, Gérard-Tahara [3] introduced the Briot-Bouquet type partial differential equation $t\partial_t u = F(t, x, u, \partial_x u)$. In [17] the author showed existences of holomorphic and singular solutions of the following type of difference-differential equations $tD_q u = F(t, x, u, \partial_x u)$ when the characteristic exponent $\rho(0) \neq (q^N - 1)/(q - 1)$ holds. In this paper the author shows existences of singular solutions with $\rho(0) = (q^N - 1)/(q - 1)$

§ 1. Introduction

In this paper let $q > 1$: for a function $f(t, x)$ we define the q -difference operator D_q by

$$D_q f(t, x) = \frac{\sigma_q f(t, x) - f(t, x)}{qt - t} = \frac{f(qt, x) - f(t, x)}{qt - t}.$$

In [15] Ramis introduced the q -difference operator D_q . We will study the following type of nonlinear q -difference-differential equations:

$$(1.1) \quad tD_q u = F(t, x, \{\partial_x^\alpha u\}_{|\alpha| \leq m})$$

where $(t, x) = (t, x_1, \dots, x_n) \in \mathbb{C}_t \times \mathbb{C}_x^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $\partial_i = \partial/\partial x_i$ for $i = 1, \dots, n$, $F(t, x, Z)$ ($Z = \{Z_\alpha\}_{|\alpha| \leq m}$) is a function defined in a polydisk Δ centered at the origin of $\mathbb{C}_t \times \mathbb{C}_x^n \times \mathbb{C}_Z^\delta$ and δ is the cardinal of $\{\alpha \in \mathbb{N}^n : |\alpha| \leq m\}$.

Received October 26, 2016. Revised January 20, 2018.

2010 Mathematics Subject Classification(s): Primary 35C10; Secondary 35C20

Key Words: q -Briot-Bouquet, Singular solutions.

*Shibaura Institute of Technology, Saitama 337-8570, Japan. e-mail: yamazawa@shibaura-it.ac.jp

In the case of q -difference equations, Vizio, Ramis, Sauloy and Zhang [2] studied linear equations and Joshi and Shi [5], Nishioka [11] and Ohyama [12] obtained some results for nonlinear equations. In the case of q -difference-differential equations, Lastra, Malek and Sanz [7] and Tahara and Yamazawa [14] studied the summability of formal solutions of linear equations.

Let us denote $\Delta_0 = \Delta \cap \{t = 0, Z = 0\}$. In this paper we assume the following conditions:

(A1) $F(t, x, Z)$ is holomorphic in Δ ,

(A2) $F(0, x, 0) = 0$ in Δ_0 ,

(A3) $\frac{\partial F}{\partial Z_\alpha}(0, x, 0) = 0$ in Δ_0 for all $1 \leq |\alpha| \leq m$.

Definition 1.1. If the equation (1.1) satisfies (A1), (A2) and (A3) we say that (1.1) is a q -analogue of the Briot-Bouquet type with respect to t (simply the q -Briot-Bouquet type with respect to t).

Definition 1.2. ([3]) Let us define

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0),$$

then the holomorphic function $\rho(x)$ is called the characteristic exponent of the equation (1.1).

Let us denote by

1. $\mathcal{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,
2. $S_\theta = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); |\arg t| < \theta\}$,
3. $S(\epsilon(s)) = \{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \epsilon(\arg t)\}$ for some positive-valued function $\epsilon(s)$ defined and continuous on \mathbb{R} ,
4. $S_\theta(T) = S_\theta \cap S(T)$,
5. $D_R = \{x \in \mathbb{C}^n; |x_i| < R \text{ for } i = 1, \dots, n\}$,
6. $\mathbb{C}\{x\}$ the ring of germs of holomorphic functions at the origin of \mathbb{C}^n ,
7. $\mathcal{O}(D)$ the set of all holomorphic functions on a domain $D \subset \mathbb{C}_x^n$,
8. $\|f\|_R := \sup_{x \in D_R} |f(x)|$.

Definition 1.3. ([3]) We define the set $\tilde{\mathcal{O}}_+$ of all functions $u(t, x)$ satisfying the following conditions:

1. $u(t, x)$ is holomorphic in $S(\epsilon(s)) \times D_R$ for some $\epsilon(s)$ and $R > 0$,
2. there is an $a > 0$ such that for any $\theta > 0$ and any compact subset K of D_R

$$\max_{x \in K} |u(t, x)| = O(|t|^a) \quad \text{as } t \rightarrow 0 \quad \text{in } S_\theta.$$

Set $\rho_q(x) = \log\{1 + (q-1)\rho(x)\}/\log q$. Then the author proved the following result:

Theorem 1.4. ([17]) *If the equation (1.1) is of the q -Briot-Bouquet type and $\rho(0) \neq [i]_q := (q^i - 1)/(q - 1)$ for $i = 1, 2, \dots$ then we have;*

(1) *(Holomorphic solutions) The equation (1.1) has a unique solution $u_0(t, x)$ holomorphic near the origin of $\mathbb{C}_t \times \mathbb{C}_x^n$ satisfying $u_0(0, x) \equiv 0$.*

(2) *(Singular solutions) When $\Re \rho(0) > 0$ for any $\varphi(x) \in \mathbb{C}\{x\}$ there exists an $\tilde{\mathcal{O}}_+$ -solution $U(\varphi)$ of (1.1) having an expansion of the following form:*

$$(1.2) \quad U(\varphi) = \sum_{i=1}^{\infty} u_i(x) t^i + \sum_{k \leq i+2m(j-1), j \geq 1} \varphi_{i,j,k}(x) t^{i+\rho_q(x)j} (\log t)^k, \quad \varphi_{0,1,0}(x) = \varphi(x).$$

The purpose of our paper is to construct $\tilde{\mathcal{O}}_+$ -solutions of (1.1) on the case $\rho(0) = [N]_q$. The main result of this paper is;

Theorem 1.5. *If (1.1) is of the q -Briot-Bouquet type and $\rho(0) \equiv [N]_q$ for $N \in \mathbb{N}$ and $\rho(x) \not\equiv \rho(0)$ then we have;*

For any $\varphi(x) \in \mathbb{C}\{x\}$ there exists an $\tilde{\mathcal{O}}_+$ -solution $U(\varphi)$ of (1.1) having an expansion of the following form:

$$(1.3) \quad U(\varphi) = \varphi(x) t^{\rho_q(x)} + \sum_{k \geq 1} \sum_{1 \leq l \leq m_k} \phi_{k,l}(x) w_{k,l}(t, x).$$

Here m_k and $w_{k,l}(t, x)$ are as follows: (1) m_k ($k \geq 1$) are positive integers determined by the equation (1.1), and (2) $w_{k,l}(t, x)$ ($k \geq 1$ and $1 \leq l \leq m_k$) are functions also determined by the equation (1.1) satisfying the following property: there is a $\sigma > 0$ such that $w_{k,l}(t, x) = O(t^{\sigma k}, \tilde{\mathcal{O}}_+)$ (as $t \rightarrow 0$) holds for all (k, l) . The coefficients $\varphi(x)$ and $\phi_{k,l}(x)$ are as follows: (3) $\varphi(x)$ are arbitrary holomorphic function, and (4) $\phi_{k,l}(x)$ ($k \geq 1$ and $1 \leq l \leq m_k$) are holomorphic functions determined by $\varphi(x)$.

In the above condition (2) the notation

$$w(t, x) = O(t^s, \tilde{\mathcal{O}}_+) \quad (\text{as } t \rightarrow 0)$$

means that the condition $t^{-s}w(t, x) \in \tilde{\mathcal{O}}_+$ holds.

This paper is organized as follows. In section 2 and 3 we give the definition and estimates of $w_{k,l}(t, x)$ in Theorem 1.5. In section 4 and 5 we prepare lemmas to show our theorem. In section 6 and 7 we give a proof of Theorem 1.5.

§ 2. Definition of the System $\{w_{k,l}\}$

In this section we will define functions $\{w_{k,l}\}$, the idea to construct the functions is in [13]. We choose a constant σ such that $0 < \sigma < \min\{1, \rho_q(0)\}$ and $\{\sigma k; k = 1, 2, \dots\} \not\subset \rho_q(0)$. Then we have an integer N^* such that

$$(2.1) \quad \sigma N^* < \rho_q(0) < \sigma(N^* + 1).$$

Remark. By the definition of $\rho_q(x)$, $\rho_q(0) > 0$ holds.

Set $\lambda(x) = 1 + (q - 1)\rho(x)$ and $a > 0$. Let us define an operator Q by

$$(2.2) \quad Q[t^N] = (\sigma_q - \lambda(x))^{-1}[t^N] := \frac{t^{\rho_q(x)} - t^{\rho_q(0)}}{\lambda(x) - \lambda(0)}$$

for $N = 1, 2, \dots$ and

$$(2.3) \quad Q[u(t, x)] := \begin{cases} \sum_{j=0}^{\infty} (\lambda(x))^j \sigma_q^{-(j+1)} u(t, x) & \text{if } \|\lambda\|_r < q^a \\ - \sum_{j=0}^{\infty} (\lambda(x))^{-j-1} \sigma_q^j u(t, x) & \text{if } \|\lambda\|_r > q^a \end{cases}$$

for $u(t, x) = O(t^a, \tilde{\mathcal{O}}_+)$

Lemma 2.1. ([17], Corollary 5.5, p.196) Suppose that $u(t, x)$ belongs to $\tilde{\mathcal{O}}_+$ and satisfies

$$\|u(t, \cdot)\|_r \leq M|t|^a \quad \text{for } t \in S_\theta(T)$$

for any $\theta > 0$ and an $a > 0$. Then there exists a positive constant C such that

$$\|Qu(t, \cdot)\|_r \leq \frac{M}{Cq^a} |t|^a \quad \text{for } t \in S_\theta(T).$$

Then we define function classes in order to construct $w_{k,l}$.

Definition 2.2. We define finite sets \mathcal{F}_k , \mathcal{G}_k and \mathcal{H}_k ($k = 1, 2, \dots$) in $\tilde{\mathcal{O}}_+$ inductively by the following procedure (1) \sim (3):

(1) We set $\mathcal{F}_1 = \{Q[t]\}$. If $k \geq 2$ and if $\mathcal{H}_1, \dots, \mathcal{H}_{k-1}$ are already defined, we set

$$\mathcal{F}_k = \bigcup_{2 \leq \mu + |\nu| \leq k} \bigcup_{\mu + k_1 + \dots + k_{|\nu|} = k} \{Q[t^\mu \phi_{k_1} \cdots \phi_{k_{|\nu|}}]; \phi_{k_j} \in \mathcal{H}_{k_j} (j = 1, \dots, |\nu|)\}.$$

(2) If \mathcal{F}_k is already defined, we set

$$\mathcal{G}_k = \begin{cases} \mathcal{F}_k & (k \neq N^*) \\ \mathcal{F}_k \cup \{t^{\rho_q(x)}\} & (k = N^*). \end{cases}$$

(3) If \mathcal{G}_k is already defined, we set

$$\mathcal{H}_k = \bigcup_{|\alpha| \leq m} \bigcup_{0 \leq \beta_\alpha \leq \alpha} \{k^{|\beta_\alpha|} \left(\frac{\partial}{\partial x}\right)^{\alpha - \beta_\alpha} W; W \in \mathcal{G}_k\}$$

where $\beta \leq \alpha$ means $\beta_j \leq \alpha_j$ for $j = 1, \dots, n$.

Definition 2.3. We define the system of functions $\{w_{k,l}(t, x) : k \geq 1, 1 \leq l \leq m_k\}$ by the following : set $m_k = \sharp \mathcal{F}_k$ and

$$\mathcal{F}_k = \{w_{k,1}(t, x), \dots, w_{k,m_k}(t, x)\} \quad \text{for } k = 1, 2, \dots$$

Then we have:

Proposition 2.4. *If $R > 0$ is sufficiently small, we have*

- (1) $w_{k,l}(t, x)$ belongs to $\tilde{\mathcal{O}}_+$ for any $k \geq 1$ and $1 \leq l \leq m_k$.
- (2) For any $\theta > 0$ there exists a $T > 0$ and $0 < \sigma' < \sigma$ such that

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha w_{k,l}(t, x) \right| \leq \frac{k^{|\alpha|}}{q^{\sigma'k}} |t|^{\sigma'k} \quad \text{on } S_\theta(T) \times D_R$$

holds for any $k \geq 1, 1 \leq l \leq m_k$ and $|\alpha| \leq m$.

In the next section, we will show the proof of Proposition 2.4.

§ 3. Proof of Proposition 2.4

In this section we will give the proof of Proposition 2.4.

Recall that $\mathcal{F}_1 = \{w_{1,1}(t, x)\}$ with $w_{1,1} = Q[t]$. Then taking $0 < \sigma < 1$ in Section 2 we can assume that $w_{1,1}$ is holomorphic on $S(1) \times D_R$ and for any $\theta > 0$ there exists $K_\theta > 1$ such that

$$(3.1) \quad \left\| \left(\frac{\partial}{\partial x}\right)^\alpha w_{1,1}(t, \cdot) \right\|_R \leq K_\theta \frac{1}{q^\sigma} |t|^\sigma \quad \text{on } S_\theta(1) \text{ for any } |\alpha| \leq m.$$

By induction on k we have:

Lemma 3.1. *For any $k = 1, 2, \dots$ we have the following properties $(1)_k$ and $(2)_k$, in which the constant $C_\theta > 1$ is independent of α, k and l .*

(1)_k $w_{k,l}(t, x)$ is holomorphic on $S_\theta(1) \times D_r$ for any $\theta > 0$ and $0 < r < R$ and $l = 1, \dots, m_k$.

(2)_k We have the following estimates for any $\theta > 0$:

$$(3.2) \quad \left\| \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t, \cdot) \right\|_r \leq \frac{k^{|\alpha|}}{q^{\sigma k}} \frac{C_\theta^{2k-2}}{(R-r)^{m(k-1)}} |t|^{\sigma k} \quad \text{on } S_\theta(1)$$

for any $0 < r < R$, $|\alpha| \leq m$ and $l = 1, \dots, m_k$.

Proof. For $t^{\rho_q(x)}$ we can assume

$$(3.3) \quad \left\| \left(\frac{\partial}{\partial x} \right)^\alpha t^{\rho_q(x)} \right\|_r \leq \frac{N^{*|\alpha|}}{q^{\sigma N^*}} \frac{C_\theta^{2N^*-2}}{(R-r)^{m(N^*-1)}} |t|^{\sigma N^*} \quad \text{on } S_\theta(1)$$

for any $0 < r < R$ and $|\alpha| \leq m$.

Let $k \geq 2$. Suppose that (1)_i and (2)_i already hold for $i = 1, \dots, k-1$. Set $\varphi_{k_\nu} = k_\nu^{|\beta_\alpha|} \left(\frac{\partial}{\partial x} \right)^{\alpha - \beta_\alpha} W$ for some $|\alpha| \leq m$, $0 \leq \beta_\alpha \leq \alpha$ and $W(t, x) \in \mathcal{G}_{k_\nu}$. Since $1 \leq k_\nu \leq k-1$ holds, in the case $W(t, x) \in \mathcal{F}_{k_\nu}$ we have

$$(3.4) \quad \begin{aligned} \|\varphi_{k_\nu}\|_r &\leq k_\nu^{|\beta_\alpha|} \frac{k_\nu^{|\alpha| - |\beta_\alpha|}}{q^{\sigma k_\nu}} \frac{C_\theta^{2k_\nu-2}}{(R-r)^{m(k_\nu-1)}} |t|^{\sigma k_\nu} \\ &\leq K_m \frac{C_\theta^{2k_\nu-2}}{(R-r)^{m(k_\nu-1)}} |t|^{\sigma k_\nu} \quad \text{on } S_\theta(1) \end{aligned}$$

for any $0 < r < R$. Here we used that there exists positive constant K_m such that

$$\frac{k^{|\alpha|}}{q^{\sigma k}} \leq K_m \quad \text{for } |\alpha| \leq m \text{ and } k = 1, 2, \dots$$

In the case $k_\nu = N^*$ and $W(t, x) = t^{\rho_q(x)}$, by the estimate (3.3) we can obtain the same estimate as (3.4). By the definition of \mathcal{F}_k , we see that $w_{k,l}(t, x)$ is expressed the form

$$w_{k,l}(t, x) = Q[t^\mu \varphi_{k_1} \times \dots \times \varphi_{k_{|\nu|}}]$$

where $2 \leq \mu + |\nu| \leq k$ and $\mu + k_1 + \dots + k_{|\nu|} = k$.

Then by (3.4) we have

$$(3.5) \quad \begin{aligned} &\|t^\mu \varphi_{k_1}(t, \cdot) \times \dots \times \varphi_{k_{|\nu|}}(t, \cdot)\|_r \\ &\leq |t|^\mu \times K_m \frac{C_\theta^{2k_1-2}}{(R-r)^{m(k_1-1)}} |t|^{\sigma k_1} \times \dots \times K_m \frac{C_\theta^{2k_{|\nu|}-2}}{(R-r)^{m(k_{|\nu|}-1)}} |t|^{\sigma k_{|\nu|}} \\ &\leq K_m^{|\nu|} \frac{C_\theta^{2(k_1 + \dots + k_{|\nu|} - |\nu|)}}{(R-r)^{m(k_1 + \dots + k_{|\nu|} - |\nu|)}} |t|^{\sigma k} \leq \frac{C_\theta^{2k-2\mu-3/2|\nu|}}{(R-r)^{m(k-\mu-|\nu|)}} |t|^{\sigma k} \quad \text{on } S_\theta(1) \end{aligned}$$

when $K_m \leq C_\theta^{1/2}$.

Since $\theta > 0$ is arbitrary, this implies that $t^\mu \varphi_{k_1} \times \cdots \times \varphi_{k_{|\nu|}}$ is holomorphic on $S(1) \times D_r$ for any $0 < r < R$. Thus we see that $w_{k,l}(t, x)$ is holomorphic on $S(1) \times D_r$, that is, $w_{k,l}(t, x)$ belongs to $\tilde{\mathcal{O}}_+$. By Lemma 2.1, we have

$$(3.6) \quad \|w_{k,l}\|_r \leq \frac{1}{Cq^{\sigma k}} \frac{C_\theta^{2k-2\mu-3/2|\nu|}}{(R-r)^{m(k-2)}} |t|^{\sigma k} \leq \frac{1}{Cq^{\sigma k}} \frac{C_\theta^{2k-3}}{(R-r)^{m(k-2)}} |t|^{\sigma k} \quad \text{on } S_\theta(1).$$

We need the following lemma in order to estimate the derivative of $w_{k,l}$.

Lemma 3.2 (Nagumo's lemma). *If a holomorphic function $u(x)$ in D_R satisfies*

$$\|u\|_r \leq \frac{C}{(R-r)^p} \quad \text{for } 0 < r < R$$

then we have

$$\left\| \frac{\partial}{\partial x} u \right\|_r \leq \frac{Ce(p+1)}{(R-r)^{p+1}} \quad \text{for } 0 < r < R.$$

For the proof, see Hörmander ([4], lemma 5.1.3).

By Lemma 3.2, we obtain

$$(3.7) \quad \begin{aligned} \left\| \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l} \right\|_r &\leq \frac{C_\theta^{2k-3}}{Cq^{\sigma k}} \frac{(m(k-2)+1)(m(k-2)+2) \cdots (m(k-2)+|\alpha|)e^{|\alpha|}}{(R-r)^{m(k-2)+|\alpha|}} |t|^{\sigma k} \\ &\leq \frac{C_\theta^{2k-3}}{Cq^{\sigma k}} \frac{(m(k-1))^{|\alpha|} e^{|\alpha|}}{(R-r)^{m(k-2)+|\alpha|}} |t|^{\sigma k} \leq \frac{C_\theta^{2k-3}(k-1)^{|\alpha|}}{Cq^{\sigma k}} \frac{(me)^m}{(R-r)^{m(k-1)}} |t|^{\sigma k} \\ &\leq \frac{k^{|\alpha|}}{q^{\sigma k}} \frac{C_\theta^{2k-2}}{(R-r)^{m(k-1)}} |t|^{\sigma k} \quad \text{on } S_\theta(1) \end{aligned}$$

for any $|\alpha| \leq m$ and $(me)^m/C \leq C_\theta$. Q.E.D.

Completion of the proof of Proposition 2.4.

Set $r = R/2$ and take $\sigma' > 0$ with $\sigma' < \sigma$ and (2.1). Then we see that $w_{k,l}(t, x)$ is holomorphic on $S(1) \times D_{R/2}$ for all (k, l) and

$$(3.8) \quad \begin{aligned} \left\| \left(\frac{\partial}{\partial x} \right)^\alpha w_{k,l}(t, \cdot) \right\|_{R/2} &\leq \frac{k^{|\alpha|}}{q^{\sigma k}} \frac{C_\theta^{2k-2}}{(R/2)^{m(k-1)}} |t|^{\sigma k} \\ &\leq \frac{k^{|\alpha|}}{q^{\sigma' k}} \left(\frac{C_\theta^2}{(R/2)^m} \left(\frac{|t|}{q} \right)^{\sigma-\sigma'} \right)^k |t|^{\sigma' k} \quad \text{on } S_\theta(1) \end{aligned}$$

holds for any $|\alpha| \leq m$ and (k, l) . Then if we take $T > 0$ so that

$$\left(\frac{C_\theta^2}{(R/2)^m} \left(\frac{|t|}{q} \right)^{\sigma-\sigma'} \right) \leq 1 \quad \text{for } |t| \leq T,$$

we obtain the estimate in Proposition 2.4-(2) with R and σ replaced $R/2$ and σ' respectively.

§ 4. Reduction equations

In this section we will reduct (1.1) into the following equation in order to prove the main theorem:

$$(4.1) \quad (\sigma_q - \lambda(x))u(t, x) = a(x)t + G_2(x)(t, \{\partial_x^\alpha u(t, x)\}_{|\alpha| \leq m})$$

where $a(x) \in \mathcal{O}(D_R)$.

Set $Z^\nu = \prod_{|\alpha| \leq m} \{Z_\alpha\}^{\nu_\alpha}$. We assume that the functions $G_2(x)(t, Z)$ have the following expansion:

$$(4.2) \quad G_2(x)(t, Z) = \sum_{\mu+|\nu| \geq 2} g_{\mu, \nu}(x) t^\mu Z^\nu$$

where $g_{\mu, \nu}(x) \in \mathcal{O}(D_R)$.

Lemma 4.1. *If the equation (1.1) is of the q -Briot-Bouquet type, then we can reduct (1.1) into the equation (4.1) with (4.2).*

Proof. We multiply the both side of (1.1) by $q - 1$. Then we get (4.1) and $\lambda(x) = 1 + (q - 1)\rho(x)$. Q.E.D.

Remark. If $\rho(0) \equiv [N]_q$ then $\lambda(0) \equiv q^N$.

§ 5. Function equation

Let us consider a functional equation in order to estimate the coefficients $w_{l, k}(x)$. Set $A \geq 0$, $G_{\mu, \nu} \geq 0$ ($\mu + |\nu| \geq 2$) and $0 < r < R$. Then we consider the following functional equation:

$$(5.1) \quad Y = At + \frac{1}{(R - r)^m} \sum_{\mu+|\nu| \geq 2} \frac{G_{\mu, \nu}}{(R - r)^{m(\mu+|\nu|-2)}} t^\mu ((2me)^m Y)^{|\nu|}$$

where a series $\sum_{\mu+|\nu| \geq 2} G_{\mu, \nu} t^\mu Y^{|\nu|}$ converges in a neighbourhood of $(t, Y) = (0, 0)$.

For the equation (5.1) we have the following lemma.

Lemma 5.1. *The functional equation (5.1) has the holomorphic solution $Y(t) = \sum_{k=1} Y_k(r) t^k$ that is expressed in the form*

$$Y_k(r) = \frac{C_k}{(R - r)^{m(k-1)}} \quad \text{for } k \geq 1$$

with constants $C_1 = A$ and $C_k \geq 0$ for $k \geq 2$.

Proof. Let us show that the equation (5.1) have formal power series solutions $Y(t) = \sum_{k=1}^{\infty} Y_k(r)t^k$. By substituting $\sum_{k=1}^{\infty} Y_k t^k$ into (5.1), we have $Y_1 = A$ and

$$(5.2) \quad Y_k = \frac{1}{(R-r)^m} \sum_{2 \leq \mu+|\nu| \leq k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \prod_{\mu+|k(\nu)|=k} \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_{\alpha}} ((2em)^m) Y_{k_{\alpha}(i)}$$

for $k \geq 2$. Let us show that the formal power series solutions $Y(t)$ are holomorphic in a neighborhood of the origin $t = 0$. Set

$$(5.3) \quad \begin{aligned} F(t, Y) &= Y - At \\ &- \frac{1}{(R-r)^m} \sum_{\mu+|\nu| \geq 2} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} t^{\mu} ((2me)^m Y)^{|\nu|}. \end{aligned}$$

We have $F(0, 0) = 0$ and $\partial_Y F(0, 0) = 1$. Then we get holomorphic solutions $Y(t)$ by the implicit function's theorem. Further we have the coefficients $Y_k(r) = C_k / (R-r)^{m(k-1)}$ by the recurrence formula (5.2) and induction on k . Q.E.D.

§ 6. Proof of Theorem 1.5 of the case $N = 1$

In this section we will give the proof of theorem 1.5. We will show that the equation (4.1) has formal solutions of the form (1.3) and the formal solution converges in $S_{\theta}(T) \times D_r$ by the implicit function's theorem and the method of majorant functions.

§ 6.1. Construction of a formal solution

We note that we have $N^* = 1$ when $N = 1$. Let us construct a formal solution $u(t, x)$ of the equation (4.1) in the form

$$(6.1) \quad u(t, x) = \sum_{k \geq 1} u_k(t, x)$$

with

$$(6.2) \quad u_k(t, x) = \begin{cases} \phi_{1,1}(x)w_{1,1}(t, x) + \varphi(x)t^{\rho_q(x)} & k = 1 \\ \sum_{l=1}^{m_k} \phi_{k,l}(x)w_{k,l}(t, x) & k \geq 2 \end{cases}$$

where $\varphi(x)$ and $\phi_{k,l}(x)$ are suitable holomorphic functions in a common neighbourhood of $x = 0$. Here set $W_{1,1} := w_{1,1}$, $W_{1,2} := t^{\rho_q(x)}$, $M_1 := m_1 + 1 = 2$, $W_{k,l} := w_{k,l}$ and $M_k := m_k$ for $k \geq 2$.

Remark. We note

$$w_{1,1}(t, x) = \frac{t^{\rho_q(x)} - t^{\rho_q(0)}}{\lambda(x) - \lambda(0)}.$$

By the proof of Proposition 2.4, we have

$$u_k(t, x) = O(t^{\sigma_k}, \tilde{\mathcal{O}}_+) \quad \text{as } t \rightarrow 0$$

for all $k \geq 1$. Then we will construct a formal solution as follows;

$$(6.3) \quad (\sigma_q - \lambda(x))u_1 = a(x)t$$

$$(\sigma_q - \lambda(x))u_k = \sum_{\mu + |k(\nu)| = k} g_{\mu, \nu}(x) t^\mu \prod_{|\alpha| \leq m} \prod_{i \leq \nu_\alpha} \left(\frac{\partial}{\partial x}\right)^\alpha u_{k_\alpha(i)}(t, x)$$

where $|\nu| = \sum_{|\alpha| \leq m} \nu_\alpha$ and $|k(\nu)| = \sum_{|\alpha| \leq m} \sum_{i=1}^{\nu_\alpha} k_\alpha(i)$. By substituting $u_k(t, x)$ to the relation (6.3), we have

$$(6.4) \quad \sum_{l=1}^{m_k} \phi_{k,l}(x) (\sigma_q - \lambda(x)) w_{k,l}(t, x)$$

$$= \sum_{2 \leq \mu + |\nu| \leq k} g_{\mu, \nu}(x) \sum_{(k(\nu), l(\nu)) \in J_k(\mu, \nu)} \sum_{\beta(\nu) \in \Gamma(\nu)} \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha}$$

$$\times \binom{\alpha}{\beta_\alpha(i)} \frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}} \left(\frac{\partial}{\partial x}\right)^{\beta_\alpha(i)} \phi_{k_\alpha(i), l_\alpha(i)}(x)$$

$$\times \left[t^\mu \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} k_\alpha(i)^{|\beta_\alpha(i)|} \left(\frac{\partial}{\partial x}\right)^{\alpha - \beta_\alpha(i)} W_{k_\alpha(i), l_\alpha(i)}(t, x) \right]$$

where $l(\nu) = \{(l_\alpha(i)); |\alpha| \leq m \text{ and } 1 \leq i \leq \nu_\alpha\}$, $L(\nu, k(\nu)) = \{l(\nu); 1 \leq l_\alpha(i) \leq m_{k_\alpha(i)} (|\alpha| \leq m, 1 \leq i \leq \nu_\alpha)\}$, $J_k(\mu, \nu) = \{(k(\nu), l(\nu)); \mu + |k(\nu)| = k, l(\nu) \in L(\nu, k(\nu))\}$, $\beta(\nu) = \{(\beta_\alpha(i)); \beta_\alpha(i) \in \mathbb{N}^n, |\alpha| \leq m \text{ and } 1 \leq i \leq \nu_\alpha\}$ and $\Gamma(\nu) = \{\beta(\nu); \beta_\alpha(i) \leq \alpha (|\alpha| \leq m, 1 \leq i \leq \nu_\alpha)\}$.

Here we note the following lemma that is the same as Lemma 8 in [13].

Lemma 6.1. *Let $k \geq 2$ and set $\mathcal{A}_k = \{(\mu, \nu, k(\nu), l(\nu), \beta(\nu)); 2 \leq \mu + |\nu| \leq k, (k(\nu), l(\nu)) \in \mathcal{J}_k(\mu, \nu) \text{ and } \beta(\nu) \in \Gamma(\nu)\}$. Then by a suitable injection $\pi_k: \mathcal{A}_k \rightarrow \{1, 2, \dots, m_k\}$ we have the following equality*

$$(6.5) \quad Q \left[t^\mu \sum_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} k_\alpha(i)^{|\beta_\alpha(i)|} \left(\frac{\partial}{\partial x}\right)^{\alpha - \beta_\alpha(i)} W_{k_\alpha(i), l_\alpha(i)}(t, x) \right] = w_{k,l}(t, x)$$

and

$$(6.6) \quad g_{\mu, \nu}(x) \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} \binom{\alpha}{\beta_\alpha(i)} \frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}} \left(\frac{\partial}{\partial x}\right)^{\beta_\alpha(i)} \phi_{k_\alpha(i), l_\alpha(i)}(x) = \phi_{k,l}(x)$$

under the correspondence $\pi_k(\mu, \nu, k(\nu), l(\nu), \beta(\nu)) = l$.

Thus we have;

Proposition 6.2. *We can construct a formal solution $\hat{u}(t, x)$ of the form (6.1) and (6.2). Moreover we see the following: (i) the coefficient $\varphi(x) \in \mathbb{C}\{x\}$ can be chosen arbitrary, (ii) $\phi_{1,1}(x) = a(x)$, and (iii) all the other coefficients $\phi_{k,l}(x) \in \mathbb{C}\{x\}$ are determined by (6.6) and therefore they are all holomorphic in a common neighborhood of $x = 0$.*

§ 6.2. Proof of the convergence of a formal solution

We will prove the convergence of the formal solution $\hat{u}(t, x)$ in Proposition 6.2. Let $R > 0$ be sufficiently small. Set

$$(6.7) \quad \begin{aligned} & \|(\frac{\partial}{\partial x})^\alpha \varphi\|_R + \|(\frac{\partial}{\partial x})^\alpha \phi_{1,1}\|_R \leq A \quad \text{for } |\alpha| \leq m \text{ and} \\ & \|g_{\mu,\nu}\|_R \leq G_{\mu,\nu} \quad \text{for } \mu + |\nu| \geq 2. \end{aligned}$$

We will assume that

$$At + \sum_{\mu+|\nu|\geq 2} G_{\mu,\nu} t^\mu Z^\nu$$

is convergent in a neighborhood of $(t, Z) = (0, 0)$.

Let $\hat{u}(t, x) = \sum_{k \geq 1} u_k(t, x)$ with

$$(6.8) \quad u_k(t, x) = \sum_{l=1}^{M_k} \phi_{k,l}(x) W_{k,l}(t, x)$$

be the formal solution. We know $M_1 = 2$, $\phi_{1,1}(x) = a(x)$, $\phi_{1,2}(x) = \varphi(x)$ and

$$\|u_k\|_r \leq \sum_{l=1}^{M_k} \|\phi_{k,l}\|_r |t|^{\sigma_k} \quad \text{for } t \in S_\theta(T).$$

Then set

$$(6.9) \quad \|u_k\|_r^* := \sum_{l=1}^{M_k} \|\phi_{k,l}\|_r \quad \text{and} \quad \|D_x^\alpha u_k\|_r^* = \sum_{l=1}^{M_k} \|(\frac{\partial}{\partial x})^\alpha \phi_{k,l}\|_r$$

in order to estimate the term $u_k(t, x)$ in (6.8).

We have $\|u_k\|_r^* = \delta_{k,1} \|\varphi\|_r + \sum_{l=1}^{m_k} \|\phi_{k,l}\|_r$ where $\delta_{k,1} = 1$ if $k = 1$, $\delta_{k,1} = 0$ if $k \neq 1$. By the relation (6.6), we obtain

$$(6.10) \quad \|u_k\|_r^* \leq \delta_{k,1} \|\varphi\|_r + \sum_{2 \leq \mu+|\nu| \leq k} \|g_{\mu,\nu}\|_r \sum_{\mu+|k(\nu)|=k, \beta(\nu) \in \Gamma(\nu)} \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} \binom{\alpha}{\beta_\alpha(i)} \\ \times \frac{1}{k_\alpha(i)^{|\beta_\alpha(i)|}} \|D_x^{\beta_\alpha(i)} u_{k_\alpha(i)}\|_r^*.$$

Let us consider the following equation:

$$(6.11) \quad Y = At + \frac{1}{(R-r)^m} \sum_{\mu+|\nu| \geq 2} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} t^\mu ((2me)^m Y)^{|\nu|}.$$

Then we have;

Proposition 6.3.

$$(6.12) \quad \frac{1}{k^{|\alpha|}} \|D_x^\alpha u_k\|_r^* \leq (me)^m Y_k \quad \text{for any } 0 < r < R \text{ and } |\alpha| \leq m,$$

for $k = 1, 2, \dots$.

Proof.

For $k = 1$ by (6.7), we have

$$(6.13) \quad \|D_x^\alpha u_1\|_r^* = \|(\frac{\partial}{\partial x})^\alpha \varphi\|_r + \|(\frac{\partial}{\partial x})^\alpha \phi_{1,1}\|_r \\ \leq A = Y_1 \leq (me)^m Y_1 \quad \text{for any } |\alpha| \leq m.$$

When $k = 1$, the inequality (6.12) holds.

For $k \geq 2$, suppose that (6.12) holds for $i = 1, 2, \dots, k-1$. Then we have

$$(6.14) \quad \|u_k\|_r^* \leq \sum_{2 \leq \mu+|\nu| \leq k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \sum_{\mu+|k(\nu)|=k, \beta(\nu) \in \Gamma(\nu)} \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} \binom{\alpha}{\beta_\alpha(i)} (me)^m Y_{k_\alpha(i)} \\ \leq \sum_{2 \leq \mu+|\nu| \leq k} \frac{G_{\mu,\nu}}{(R-r)^{m(\mu+|\nu|-2)}} \sum_{\mu+|k(\nu)|=k} \prod_{|\alpha| \leq m} \prod_{i=1}^{\nu_\alpha} (2me)^m Y_{k_\alpha(i)} \\ = (R-r)^m Y_k = \frac{C_k}{(R-r)^{m(k-2)}}.$$

Here we used

$$\sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} = 2^{|\alpha|} \leq 2^m.$$

By applying Lemma 3.2 to the inequality (6.14), we obtain

$$\begin{aligned}
 (6.15) \quad \frac{1}{k^{|\alpha|}} \|D_x^\alpha u_k\|_r^* &\leq \frac{1}{k^{|\alpha|}} \frac{(m(k-2)+1) \times \cdots \times (m(k-2)+|\alpha|) e^{|\alpha|} C_k}{(R-r)^{m(k-2)+|\alpha|}} \\
 &\leq \frac{(k-1)^{|\alpha|}}{k^{|\alpha|}} \frac{(me)^{|\alpha|} C_k}{(R-r)^{m(k-2)+|\alpha|}} \leq (me)^m \frac{C_k}{(R-r)^{m(k-1)}} = (me)^m Y_k.
 \end{aligned}$$

Q.E.D.

Proof of the convergence of the formal solution $\hat{u}(t, x)$. Set $r = R/2$. By Proposition 2.4-(2), 6.3, (6.8) and (6.9), we have

$$|u(t, x)| \leq \sum_{k \geq 1} \|u_k(t, \cdot)\|_r \leq \sum_{k \geq 1} \|u_k\|_r^* |t|^{\sigma k} \leq (me)^m \sum_{k \geq 1} Y_k |t|^{\sigma k} \quad \text{on } S_\theta(T).$$

$u(t, x)$ converges on $S_\theta(T) \times D_r$. Since $\theta > 0$ is arbitrary, we can conclude that $u(t, x)$ converges in $\tilde{\mathcal{O}}_+$.

§ 7. Proof of Theorem 1.5 of the case $N \geq 2$

We set

$$(7.1) \quad u(t, x) = \sum_{i=1}^{N-1} u_i(x) t^i + t^{N-1} w(t, x)$$

where $u_i(x) \in \mathbb{C}\{x\}$ ($1 \leq i \leq N-1$) and $w(t, x) \in \tilde{\mathcal{O}}_+$. Then by an easy calculation we see;

Lemma 7.1. *If the function (7.1) is a solution of the equation (4.1), then the functions $u_1(x), \dots, u_{N-1}(x)$ are uniquely determined and $w(t, x)$ satisfies the following equation:*

$$\begin{aligned}
 (7.2) \quad (\sigma_q - q^{-(N-1)} \lambda(x)) w &= t a(t, x) + t \sum_{|\alpha| \leq m} A_\alpha(t, x) \partial_x^\alpha w \\
 &+ \sum_{|\nu| \geq 2} t^{(N-1)(|\nu|-1)} B_\nu(t, x) \prod_{|\alpha| \leq m} \{\partial_x^\alpha w\}^{\nu_\alpha},
 \end{aligned}$$

where

$$a(t, x) = \frac{1}{q^{N-1} t^N} (G_2(x)(t, \{\partial_x^\alpha w_0\}_{|\alpha| \leq m}) + t a(x) - (\sigma_q - \lambda(x)) w_0)$$

with $w_0 = \sum_{i=1}^{N-1} u_i(x)t^i$ and

$$A_\alpha(t, x) = \frac{1}{q^{N-1}t} \frac{\partial G_2}{\partial Z_\alpha}(x)(t, \{\partial_x^\alpha w_0\}_{|\alpha| \leq m}), \quad |\alpha| \leq m$$

$$B_\nu(t, x) = \frac{1}{q^{N-1}} \frac{1}{\nu!} \frac{\partial^{|\nu|} G_2}{\partial Z^\nu}(x)(t, \{\partial_x^\alpha w_0\}_{|\alpha| \leq m}), \quad |\nu| \geq 2.$$

We can apply the results in Section 6 to the equation (7.2). Hence this completes the proof of Theorem 1.5.

Acknowledgement. The author would like to express thanks to the referee for his comments.

References

- [1] Briot, Ch. and Bouquet, J. Cl., *Recherches sur les propriétés des fonctions définies par des équations différentielles*, J. Ecole Polytech., 21(1856), 133–197.
- [2] Vizio, L. D., Ramis, J. P., Sauloy, J. and Zhang, C., *Équations aux q -différences*, (French), Gaz. Math. No. 96(2003), 20–49.
- [3] Gérard, R. and Tahara, H., *Holomorphic and Singular Solutions of Nonlinear Singular First Order Partial Differential Equations*, Publ. RIMS, Kyoto Univ., 26(1990), 979–1000.
- [4] Hörmander, L., *Linear partial differential operators*, Springer, 1963.
- [5] Joshi, N. and Shi, Y., *Exact solutions of a q -discrete second Painlevé equation from its isomonodromy deformation problem. II. Hypergeometric solutions*, Proc. R. Lond. Ser. A Math. Phys. Eng. Sci., 468(2012), No. 2146, 3247–3264.
- [6] Lastra, A. and Malek, S., *On q -Gevrey asymptotics for singularly perturbed q -difference-differential problems with an irregular singularity*, Abstr. Appl. Anal. (2012), Art. ID 860716, 35 pp
- [7] Lastra, A., Malek, S. and Sanz, J., *On q -asymptotics for linear q -difference-differential equations with Fuchsian and irregular singularities*, J. Differential Equations 252(2012), No. 10, 5185–5216.
- [8] Malek, S., *On complex singularity analysis for linear q -difference-differential equations*, J. Dyn. Control Syst. 15(2009), No. 1, 83–98.
- [9] Malek, S., *On singularly perturbed q -difference-differential equations with irregular singularity*, J. Dyn. Control Syst. 17(2011), No. 2, 243–271.
- [10] Menous, F., *An example of local analytic q -difference equation: analytic classification*, Ann. Fac. Sci. Toulouse Math. (6) 15(2006), No. 4, 773–814.
- [11] Nishioka, S., *On solutions of q -Painlevé equation of type $A(1)7$* , Funkcial Ekvac. 52(2009), No. 1, 41–51.
- [12] Ohshima, Y., *Expansion on special solutions of the first q -Painlevé equation around the infinity*, Proc. Japan Acad. Ser. A Math. Sci. 86(2010), No. 5, 91–92.
- [13] Tahara, H. and Yamazawa H., *Structure of solutions of nonlinear partial differential equations of Gérard-Tahara type*, Publ. R.I.M.S, kyoto Univ., Vol. 41(2005), No. 2, 339–373.
- [14] Tahara, H. and Yamazawa H., *q -Analogue of summability of formal solutions of some linear q -difference-differential equations*, J. Opuscula Math., Vol. 35, no. 5 (2015), 713–738.

- [15] Ramis J. P., *About the growth of entire functions solutions of linear algebraic q -difference equations*, Annal. de la Faculté des sci. de Toulouse, Série 6, Vol. 1(1992), No. 1, 53– 94.
- [16] Yamazawa H., *Singular Solutions of the Briot-Bouquet Type Partial Differential Equations*, J. Math. Soc. Japan, Vol. 55(2003), No. 3, 617–632.
- [17] Yamazawa H., *Holomorphic and singular solutions of q -difference-differential equations of Briot-Bouquet type*, Funkcial Ekvac., Vol. 59(2016), 185–197.